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Article

Wave propagation and soliton behavior in biomechanical tissues: A mathematical approach

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Abstract: This study presents a mathematical model for understanding wave propagation and soliton behavior in biomechanical tissues, explicitly focusing on the Achilles tendon. Utilizing the Korteweg-de Vries (KdV) equation, the research incorporates the Achilles tendons' nonlinear elastic and viscoelastic properties to explore how mechanical waves propagate through this complex tissue. The tendon's nonlinear elasticity leads to wave steepening, while its viscoelasticity introduces dispersive effects that counteract this steepening, resulting in the formation of solitons—stable, localized waves that maintain their shape as they propagate. Key findings from this study reveal that the formation and propagation of solitons are strongly influenced by the tendon's mechanical properties. Numerical simulations show that stiffer tendons, characterized by a higher elasticity modulus, support faster soliton propagation, with wave speeds ranging from 18.9 m/s in damaged tendons to 28.6 m/s in stiffened tendons. Additionally, soliton amplitude increases with tissue stiffness, with the highest amplitude observed in stiffened tendons (5.1 mm) and the lowest in damaged tendons (3.2 mm). The study also demonstrates that energy dissipation due to the tendon's viscoelasticity plays a critical role in soliton behavior. Damaged tendons exhibit the highest energy loss (18.6%), leading to shorter soliton propagation distances, while stiffer tendons retain more energy (96.1%) and allow solitons to travel further distances (up to 180 mm). Moreover, the balance between nonlinearity and dispersion is crucial for maintaining soliton stability. Excessive nonlinearity leads to unstable solitons, while higher levels of dispersion contribute to more stable waveforms.

Keywords: biomechanical tissues; tendon's viscoelasticity; soliton propagation distances; mechanical waves; Achilles tendon

1. Introduction

The propagation of mechanical waves through biological tissues is a critical aspect of biomechanics, with wide-ranging implications for understanding how forces are transmitted, distributed, and absorbed in the human body [1,2]. Tissues such as tendons, muscles, and ligaments undergo continuous stress and deformation during walking, running, and jumping [3]. Among these tissues, the Achilles tendon (**Figure 1**) plays a vital role in locomotion, acting as a bridge between the calf muscles and the heel bone [4]. The unique biomechanical properties of the Achilles tendon—its elasticity, viscoelasticity, and capacity to withstand high loads—make it an ideal candidate for studying wave propagation dynamics [5]. In particular, this research focuses on modelling soliton behavior in the Achilles tendon, which arises from the balance between nonlinear and dispersive effects during wave propagation.

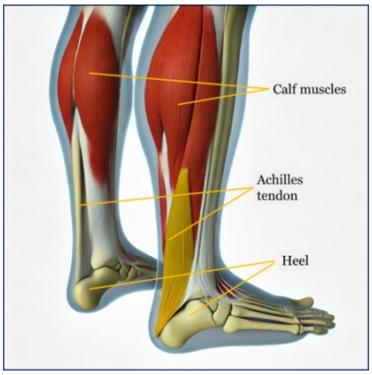

Figure 1. Achilles tendon.

The nonlinear elastic properties of biological tissues result from their complex internal structure [6]. Like other tendons, the Achilles tendon exhibits a nonlinear response to mechanical loading, especially at higher strain rates, where the tissue stiffens as it is stretched. This nonlinear stiffening leads to wave steepening, which can result in the formation of solitons—stable, localized waveforms that propagate without changing shape [7,8]. Solitons are well-known in other physical systems, such as fluid dynamics and plasma physics, but their role in biomechanical tissues is still being explored [9]. Soliton theory provides a promising framework for understanding how mechanical signals are transmitted in the Achilles tendon, where waves generated by muscle contractions or external forces can travel efficiently without dissipating energy [10].

Dispersion is another critical factor influencing the Achilles tendon wave behaviour [11]. The tendon's viscoelastic nature introduces dispersive effects, meaning that waves of different frequencies propagate at different speeds [12]. These dispersive effects tend to spread the wave over time, counteracting the steepening caused by nonlinearity [13]. The delicate balance between nonlinearity (which steepens the wave) and dispersion (which spreads the wave) results in the formation of solitons [14]. The Korteweg-de Vries (KdV) equation is widely used to model these phenomena in various media, and this study extends its application to the Achilles tendon to describe the propagation of mechanical waves and solitons [15,16].

In recent years, the study of wave propagation in biological tissues has gained attention, particularly in the context of biomedical engineering and rehabilitation sciences [17–20]. Understanding how waves travel through tendons could lead to new insights into injury mechanisms, rehabilitation protocols, and diagnostic techniques [21–23]. For instance, changes in the mechanical properties of the Achilles tendon—such as those caused by age, injury, or overuse—could alter wave behavior, impacting how forces are transmitted during movement [24–28]. By modelling the Achilles tendon as a nonlinear, dispersive medium, this research aims to shed light on the fundamental principles governing mechanical wave transmission in tendons. The proposed work aims to model and analyze wave propagation and soliton behavior in biomechanical tissues, explicitly focusing on the Achilles tendon. Utilizing a combination of nonlinear wave theory and dispersion modelling, this study extends the classical Korteweg-de Vries (KdV) equation to incorporate both the nonlinear elastic response of the tendon and the dispersive effects resulting from its viscoelastic properties. The dispersive modelling is crucial to account for the spreading of wave components of different frequencies, which counterbalances the wave steepening caused by nonlinearity, leading to the formation of stable solitons. By deriving and solving the governing equations for soliton formation, the research explores the conditions under which these localized solitons can form and propagate through the tendon [29–32]. The study includes analytical solutions for idealized cases and numerical simulations to capture complex tissue behavior, comprehensively analysing how tissue stiffness, density, and viscoelasticity affect wave dynamics.

The objectives of this study are threefold.

- First, we aim to derive the mathematical equations that govern wave propagation in the Achilles tendon, considering its nonlinear and dispersive properties. The derivation will extend the classical wave equation to include the effects of nonlinear elasticity and viscoelastic dispersion, resulting in the KdV equation.
- Second, we will investigate the conditions under which solitons form in the Achilles tendon by solving the KdV equation analytically and numerically. The soliton solutions will provide insights into how waves behave under different mechanical conditions, such as varying tissue stiffness and damping.
- Finally, we will analyze how tissue properties—such as elasticity, density, and viscosity—affect soliton behavior, focusing on key parameters like soliton amplitude, speed, and stability. These findings could have implications for biomechanical modelling, rehabilitation strategies, and diagnostic techniques for tendon injuries.

The paper is presented as follows: Section 2 presents the materials and methods, Section 4 details the mathematical model for wave propagation in biomechanical tissues, Section 5 presents the soliton behavior in biomechanical tissues, Section 6 presents the results, and Section 7 concludes the paper.

2. Materials and methods

2.1. Mathematical framework

This section describes the mathematical foundation used to model wave propagation and soliton behavior in the Achilles tendon. Given the tendon's elastic and viscoelastic properties, a nonlinear wave model is appropriate to capture the dynamics of mechanical waves propagating through the tissue [33–37].

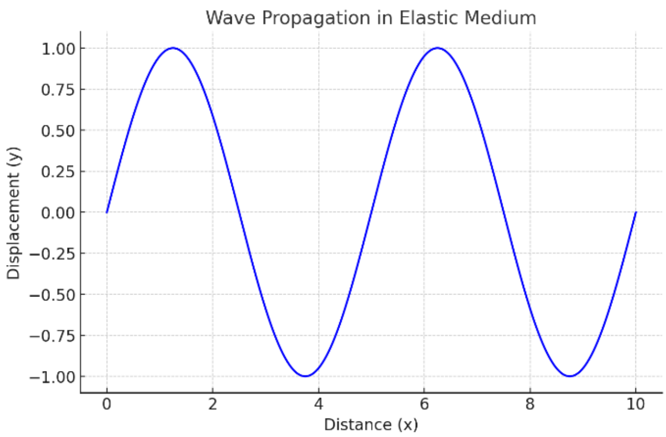

Figure 2. Wave propagation in an elastic medium.

2.1.1. Wave propagation in biomechanical tissues

The Achilles tendon is an elastic structure, meaning it deforms in response to stress and returns to its original state upon removal of the stress, allowing us to model it as an elastic medium (**Figure 2**). To describe the wave propagation in this medium start with the general one-dimensional wave equation in a homogeneous medium:

$$\frac{\partial^2 \mathrm{u}}{\partial \mathrm{t}^2} = \mathrm{c}^2 \frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2} \tag{1}$$

where $\mathrm{u}(x, t)$ represents the displacement of a point in the tendon as a function of position x and time t, c is the wave speed, which depends on the elastic properties of the tendon. However, this linear wave equation is insufficient to capture the full complexity of the Achilles tendon, mainly because it does not account for nonlinearity or dispersion. To address this, we introduce nonlinearity through the Korteweg-de Vries (KdV) equation, widely used in modeling solitons in nonlinear media.

2.1.2. Nonlinear wave equation: Korteweg-de Vries (KdV) equation

To model the wave propagation in the Achilles tendon, we extend the wave equation to include both nonlinear and dispersive effects. This leads us to the KdV equation:

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} + \alpha \mathrm{u} \frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \beta \frac{\partial^3 \mathrm{u}}{\partial \mathrm{x}^3} = 0 \tag{2}$$

where $\mathrm{u}(x, t)$ is again the displacement field of the Achilles tendon, α is a constant that quantifies the nonlinearity of the medium (representing the tendon's nonlinear elastic response), β is a dispersion constant that accounts for the tendon's viscoelastic nature, leading to wave dispersion.

The KdV equation captures both the nonlinear steepening of waves and the dispersive effects that prevent wave breaking, resulting in the formation of solitons. These solitons are stable, localized waveforms that propagate without changing shape, which makes them suitable models for the mechanical wave behavior in tendons.

- Nonlinear Effects in the Achilles Tendon: The term $\alpha u \frac{\partial u}{\partial x}$ in the KdV equation accounts for nonlinear effects. In the Achilles tendon, nonlinearity arises because the tendon does not respond to stress linearly, particularly at higher strain rates, as in athletic activities. The nonlinear term reflects the elastic stiffening that occurs as the tendon stretches. This results in the waves propagating faster in regions of higher displacement, leading to steepening of the wavefronts.
- Dispersive Effects: The dispersive term $\beta \frac{\partial^3 u}{\partial x^3}$ Represents the dispersive effects that occur in the tendon due to its viscoelastic properties. Dispersive effects cause waves of different frequencies to travel at different speeds, spreading the wave out over time. In the Achilles tendon, these effects are critical to prevent wave singularities and ensure that the waves maintain their form, a characteristic of solitons.

2.1.3. Incorporating boundary conditions

For the Achilles tendon, which is anchored at both ends (to the calf muscles and the heel), we assume fixed boundary conditions:

$$u(0,t) = u(L,t) = 0 \tag{3}$$

where L is the length of the tendon. These boundary conditions ensure that the displacement at the tendon's attachment points remains zero, corresponding to the physical constraints of the muscle tendon-bone system.

We assume that at time $t = 0$, the tendon is subjected to a localized disturbance, which can be represented as:

$$u(x,0) = f(x),\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) \tag{4}$$

where $f(x)$ and $g(x)$ are initial displacement and velocity profiles, respectively.

These initial conditions can be tailored to represent a specific impact or mechanical stimulus applied to the Achilles tendon.

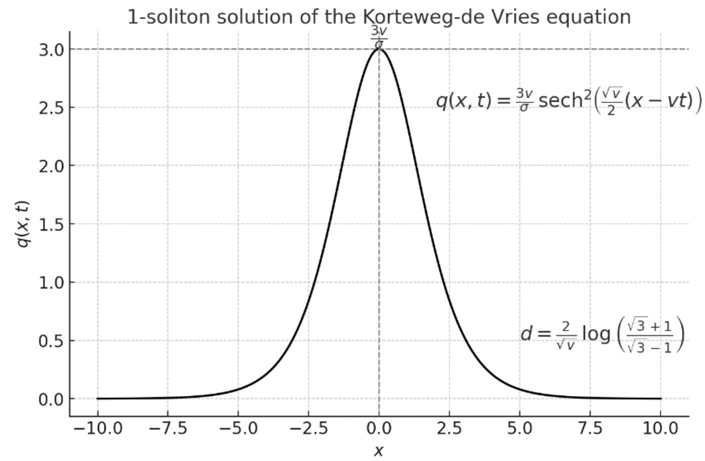

Figure 3. 1-soliton solution to the KdV.

2.1.4. Soliton solutions

Soliton solutions arise when the nonlinear and dispersive terms in the KdV equation balance. For the Achilles tendon, we expect localized solitons to form under certain conditions, and the simplest soliton solution to the KdV equation (**Figure 3**) is given by:

$$u(x,t) = A\mathrm{sech}^2\left(\frac{x - vt}{\Delta}\right) \tag{5}$$

where A is the amplitude of the soliton, v is the velocity of the soliton, Δ is the width of the soliton. This solution describes a stable, localized wave propagating through the Achilles tendon without changing shape, representing a mechanical signal travelling through the tendon.

2.2. Mechanical properties of tissues

The mechanical properties of the Achilles tendon play a crucial role in determining the characteristics of wave propagation and soliton behavior. This section describes the relevant tissue parameters and assumptions used to model the tendon based on a literature review and experimental data.

2.2.1. Elasticity modulus

The elastic modulus (also known as Young's modulus, E) quantifies the stiffness of the Achilles tendon, representing its ability to resist deformation under load (**Figure 4**). For the Achilles tendon, the elastic modulus can vary depending on factors such as age, activity level, and injury status. In healthy adults, values for the Achilles tendons elastic modulus are typically reported in the range of 500–1500 MPa based on experimental tensile tests and ultrasound elastography measurements.

$$E = \frac{\sigma}{\varepsilon} \tag{6}$$

where σ is the stress applied to the tendon (force per unit area), and ε is the strain (the relative deformation of the tendon). Higher E values correspond to a stiffer tendon, resulting in faster wave propagation. The Achilles tendon is stiffer when under high loads, such as running or jumping, which influences the formation of nonlinear waveforms, including solitons.

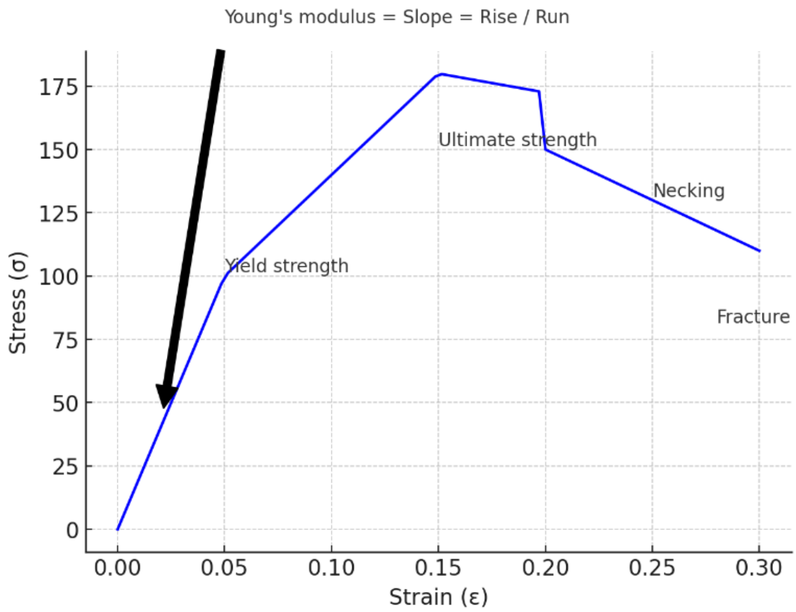

Figure 4. Young's modulus.

2.2.2. Density

The density of the Achilles tendon, denoted by ρ, affects the inertia of the tissue and thus influences wave propagation speed. Experimental studies report that the density of tendon tissue is approximately 1.12 g/cm^3. This value is critical for determining the speed of longitudinal waves, as the wave speed c depends on both the elastic modulus E and the density ρ according to:

$$c = \sqrt{\frac{E}{\rho}} \tag{7}$$

The Achilles tendon's density is relatively constant across individuals, meaning that variations in wave speed are primarily attributed to changes in elasticity or tissue heterogeneity rather than density fluctuations.

2.2.3. Viscoelasticity

The Achilles tendon exhibits viscoelastic properties, meaning its mechanical response includes both elastic (immediate) and viscous (time-dependent) components. Viscoelasticity is vital in modelling wave dispersion in the tendon. The viscoelastic nature of the tendon means that waves do not propagate as purely elastic waves but instead experience energy dissipation, leading to the spreading out of the wave over time. A constitutive equation that combines elastic and viscous effects can represent the viscoelastic behaviour. A commonly used model is the Kelvin-Voigt model, where stress σ is related to both strain ε and strain rate $\dot{\varepsilon}$:

$$\sigma = E\varepsilon + \eta\dot{\varepsilon} \tag{8}$$

where, η is the viscosity of the tissue, representing the rate-dependent behavior, $\dot{\varepsilon}$ is the time derivative of strain. The viscosity η in the Achilles tendon has been experimentally measured to vary depending on activity but typically ranges from 10^3–10^4 Pa\cdotps. This term introduces damping effects in wave propagation, affecting both the amplitude and speed of the waves and contributing to the overall stability of solitons.

2.3. Assumptions about the Achilles tendon

In constructing the mathematical model for wave propagation in the Achilles tendon, several simplifying assumptions are made regarding its material properties:

- Homogeneous and Isotropic: The Achilles tendon is assumed to be homogeneous, meaning its mechanical properties are uniform throughout the tissue. While tendons are composed of collagen fibers, for this model, we assume that these fibers contribute to a uniformly distributed stiffness. The tendon is also treated as isotropic, meaning its mechanical properties are the same in all directions, though, in reality, tendons have a degree of anisotropy due to the alignment of collagen fibers.
- Linear Elasticity at Small Strains: For small deformations, the Achilles tendon is treated as a linear elastic material, meaning that stress and strain are linearly related (i.e., Hookean behavior). This assumption holds for small displacements, but nonlinearity becomes more pronounced at higher strains, which is accounted for in the nonlinear terms of the KdV equation used later in the modeling process.
- Viscoelastic Effects Included: The viscoelastic nature of the Achilles tendon is incorporated into the model to capture both the tissue's elastic and time-dependent (viscous) response. The Kelvin-Voigt model represents this behavior, capturing the dissipation and energy loss as waves propagate through the tendon.
- Constant Properties: The model assumes that the mechanical properties of the tendon (elasticity, density, and viscosity) remain constant during wave propagation. In reality, properties such as elasticity can vary with strain and loading conditions, but these variations are assumed to be small enough not to affect the overall wave dynamics in this study significantly.

2.4. Derivation of wave equations

The mathematical model for wave propagation in the Achilles tendon builds on the foundational principles of continuum mechanics, where the tendon is treated as an elastic medium capable of supporting different types of mechanical waves. In this section, we derive the wave equations for the Achilles tendon, considering both longitudinal and transverse waves, which reflect the complex motion that occurs when the tendon is subjected to external forces.

2.4.1. Wave propagation in elastic media

In a general elastic medium like the Achilles tendon, Newton's second law of motion can describe the motion of particles within the tissue. The displacement field, $\mathrm{u}(x,t)$, represents the displacement of points within the tendon as a function of position x and time t. To derive the governing equation for wave propagation, we start with the balance of linear momentum,

Lemma 1. *Momentum Balance: The balance of linear momentum gives the equation of motion for a point within a continuous medium:*

$$\rho\frac{\partial^2 \mathrm{u}}{\partial \mathrm{t}^2}=\frac{\partial \sigma}{\partial \mathrm{x}} \tag{9}$$

where, ρ is the density of the Achilles tendon, $u(x,t)$ is the displacement in the direction of the wave, and σ is the stress experienced by the tendon.

Proof of Lemma 1. This follows directly from Newton's second law of motion for a continuous medium. The mass per unit volume (or density ρ) multiplied by the acceleration $\frac{\partial^2 u}{\partial t^2}$ gives the inertial force per unit volume. The stress gradient $\frac{\partial \sigma}{\partial x}$ represents the internal forces acting on the tendon. □

Lemma 2. *Linear Stress-Strain Relationship (Hooke's Law): To describe how stress σ and strain ε are related in the Achilles tendon, we assume that the tendon behaves according to Hooke's law in the elastic regime:*

$$\sigma=\mathrm{E}\varepsilon=\mathrm{E}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} \tag{10}$$

where E is the Young's modulus of the tendon, and $\varepsilon=\frac{\partial u}{\partial x}$ is the strain, representing the relative displacement of particles in the tendon.

Proof of Lemma 2. Hooke's law for one-dimensional elasticity in homogeneous, isotropic medium states that stress is proportional to strain, which is valid for small deformations. In this case, the strain is simply the spatial derivative of the displacement field. □

Theorem 1. *Longitudinal Wave Equation: Substituting the stress-strain relation from **Lemma 2** into the momentum balance equation from **Lemma 1**, we obtain:*

$$\rho\frac{\partial^2 \mathrm{u}}{\partial \mathrm{t}^2}=\mathrm{E}\frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2} \tag{11}$$

Rearranging this equation yields the one-dimensional wave equation:

$$\frac{\partial^2 \mathrm{u}}{\partial \mathrm{t}^2}=\mathrm{c}^2\frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2} \tag{12}$$

where $c = \sqrt{\frac{E}{\rho}}$ is the wave speed, which depends on both the elasticity modulus E and the density ρ of the tendon. This equation describes the propagation of longitudinal waves involving particle motion in the same direction as the wave. These waves typically arise when the tendon is stretched or compressed along its length.

2.4.2. Nonlinear extension for the Achilles tendon

While the linear wave equation provides a basic model for wave propagation, the Achilles tendon exhibits nonlinear behavior, especially under high-stress conditions. To account for this, we introduce a nonlinear term into the wave equation to represent the nonlinear elasticity of the tendon at higher strains.

Lemma 3. *Nonlinear Stress-Strain Relationship: At higher strains, the stress-strain relationship can be extended by including a nonlinear term, leading to:*

$$\sigma = E\frac{\partial u}{\partial x} + \alpha u\frac{\partial u}{\partial x} \tag{13}$$

where α is a constant that quantifies the degree of nonlinearity in the tissue.

Proof of Lemma 3. This nonlinear extension can be justified using a Taylor series expansion of the stress-strain relationship around minor strains, where higher-order terms account for nonlinear effects. The second term introduces a nonlinear dependence on the displacement field $u(x,t)$, which becomes significant at higher strains. □

Theorem 2. *Nonlinear Longitudinal Wave Equation: Substituting the nonlinear stress-strain relation from Lemma 3 into the momentum balance equation, we obtain the nonlinear wave equation:*

$$\rho\frac{\partial^2 u}{\partial t^2} = E\frac{\partial^2 u}{\partial x^2} + \alpha\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) \tag{14}$$

Simplifying:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + \alpha u\frac{\partial u}{\partial x} \tag{15}$$

This equation captures linear wave propagation (the first term) and the nonlinear effects (the second term), with α controlling the nonlinearity.

Proof of Theorem 2. The derivation follows from the nonlinear stress-strain relationship and its substitution into the momentum balance equation, yielding a nonlinear extension to the standard wave equation. □

2.4.3. Transverse waves

In addition to longitudinal waves, the Achilles tendon can also support transverse waves, where particle motion occurs perpendicular to the direction of wave propagation. These waves typically arise when the tendon is subjected to lateral forces, such as twisting or bending motions. The governing equation for transverse wave propagation is similar to the longitudinal wave equation but involves a different restoring force, which depends on the tendon's shear modulus G.

Lemma 4. *Transverse Stress-Strain Relationship: For transverse waves, the stress-strain relationship involves the shear modulus G, which relates shear stress to shear strain. The transverse displacement $v(x,t)$ follows:*

$$\sigma_T = G\frac{\partial v}{\partial x} \tag{16}$$

where G is the shear modulus, $\frac{\partial v}{\partial x}$ *is the shear strain.*

Proof of Lemma 4. This is the shear equivalent of Hooke's law for longitudinal waves, where stress is proportional to strain, but here, the relevant modulus is the shear modulus GGG, which governs the material's resistance to shear deformation. □

Theorem 3. *Transverse Wave Equation: Using the momentum balance equation for transverse displacements, we have:*

$$\rho\frac{\partial^2 v}{\partial t^2} = G\frac{\partial^2 v}{\partial x^2} \tag{17}$$

Rearranging, we obtain the transverse wave equation:

$$\frac{\partial^2 v}{\partial t^2} = c_T^2\frac{\partial^2 v}{\partial x^2} \tag{18}$$

where $c_T = \sqrt{\frac{G}{\rho}}$ *is the speed of transverse waves in the Achilles tendon.*

Proof of Theorem 3. This equation is derived using the shear stress-strain relationship and the momentum balance for transverse displacements, yielding a wave equation similar to the longitudinal case but involving the shear modulus G instead of the elastic modulus E. □

2.5. Influence of tissue properties on wave behavior

The behavior of waves propagating through the Achilles tendon is profoundly influenced by its mechanical properties, particularly its stiffness (elastic modulus), density, and viscoelastic nature. These properties dictate the speed at which waves travel, the frequencies they can sustain, and the degree of dispersion they experience. In this section, we explore how these factors affect wave propagation in the Achilles tendon and how they are modelled mathematically.

2.5.1. Wave speed and tissue stiffness

The stiffness of the Achilles tendon is primarily quantified by its elastic modulus E, which defines how resistant the tissue is to deformation under mechanical load. The relationship between wave speed c and the elastic modulus can be derived from the linear wave equation for longitudinal waves:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \tag{19}$$

where $c = \sqrt{\frac{E}{\rho}}$. This equation shows that wave propagation speed is directly proportional to the square root of the elastic modulus E and inversely proportional to the square root of the density ρ.

For example, the wave speed will increase if the Achilles tendon becomes stiffer (i.e., higher E). This happens because stiffer materials resist deformation more vigorously, causing mechanical waves to travel faster through the tissue. Conversely, if the density ρ increases (e.g., due to higher mass or fluid content in the tissue), wave speed decreases as it becomes more challenging for the waves to propagate through the denser medium.

2.5.2. Influence of density on wave propagation

As seen in the relationship $c = \sqrt{\frac{E}{\rho}}$, the density ρ of the Achilles tendon plays an inverse role in determining wave speed. Tendons are relatively uniform in density, but minor variations can occur due to changes in hydration levels or microscopic tissue composition (e.g., collagen fiber orientation). If the density of the tendon increases, the wave speed decreases, as the increased mass requires more energy to propagate the wave.

While variations in density are typically more minor than variations in stiffness, the overall wave propagation behavior is still influenced by this factor. For instance, localized swelling or microstructural changes in the tendon can slightly alter the density, which could impact how mechanical waves travel through the affected area.

2.5.3. Frequency and wavelength of propagating waves

The frequency f of waves in the Achilles tendon is related to the wave speed c and the wavelength λ by the equation:

$$c = f\lambda \tag{20}$$

For a given frequency, higher wave speeds (due to higher stiffness or lower density) result in longer wavelengths. In contrast, lower wave speeds result in shorter wavelengths. The frequency of waves propagating through the tendon is determined by the nature of the mechanical loading (e.g., the frequency of muscle contractions during walking or running). Low-frequency waves correspond to larger-scale deformations, while high-frequency waves correspond to finer mechanical details. High-frequency waves tend to experience more dispersion (i.e., their wave components spread out as they travel), particularly in tissues with significant viscoelasticity, such as the Achilles tendon.

2.5.4. Dispersion effects and tissue viscoelasticity

The Achilles tendon is not a purely elastic material; it exhibits viscoelastic properties, meaning that both elastic and viscous (time-dependent) forces contribute to its mechanical response. This viscoelasticity causes dispersion, which refers to the phenomenon where waves of different frequencies travel at different speeds. In viscoelastic materials, higher-frequency waves are typically more dispersed than lower-frequency waves. The general equation governing dispersive wave propagation can be written as:

$$\frac{\partial^2 u}{\partial t^2} + \eta \frac{\partial^3 u}{\partial x^3} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{21}$$

here, η is a coefficient that accounts for the tendon's viscoelastic damping, and the third derivative term $\frac{\partial^3 u}{\partial x^3}$ Introduces dispersion. The viscoelastic nature of the tendon means that waves do not propagate at a single, uniform speed; instead, higher-frequency components are delayed more than lower-frequency ones, leading to the spreading or "smearing" of the wave as it propagates.

Dispersion is significant in studying solitons—nonlinear waves that balance the effects of dispersion and nonlinearity. In the Achilles tendon, solitons can form when the nonlinear effects of wave steepening (due to tissue stiffness) are balanced by the dispersive effects of viscoelasticity, allowing the wave to travel without changing shape.

2.5.5. Modeling dispersion and nonlinearity

To model the dispersive and nonlinear behavior in the Achilles tendon, we use the Korteweg-de Vries (KdV) equation:

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} + \alpha \mathrm{u}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \beta\frac{\partial^3 \mathrm{u}}{\partial \mathrm{x}^3} = 0 \tag{22}$$

where α represents the nonlinear steepening term (related to tissue stiffness), and β represents the dispersion term (related to tissue viscoelasticity). The term $\alpha u \frac{\partial u}{\partial x}$ models the nonlinear effects caused by the tendon's elasticity, while $\beta \frac{\partial^3 u}{\partial x^3}$ Models the dispersive effects due to viscoelastic damping. Together, these terms describe how waves propagate in the tendon, including the formation of stable soliton solutions.

3. Nonlinear wave propagation

In the context of the Achilles tendon, nonlinear wave propagation occurs when the tissue experiences sufficiently high strains such that its response to mechanical waves deviates from linear elasticity. The nonlinear behavior arises due to the inherent mechanical properties of the tendon, particularly its ability to stiffen as it stretches. In this section, we explore the conditions under which nonlinear effects dominate and derive the relevant nonlinear equations that describe soliton behavior in the tendon.

Nonlinear effects become significant in the Achilles tendon when the magnitude of the deformation is large enough that the linear approximation of stress and strain is no longer valid. This typically occurs under high mechanical loads, such as during intense physical activities like running, jumping, or rapid changes in direction. In these cases, the Achilles tendon stretches beyond its elastic limits, causing a nonlinear response in its mechanical properties.

Several key factors govern the dominance of nonlinear effects:

High Strain Levels: As strain increases, the tendon stiffens nonlinearly, causing the relationship between stress and strain to deviate from Hooke's law. At low strain levels, the linear approximation holds, but at higher strains, the stiffness of the tendon increases rapidly.

Tissue Stiffness: Tendons with higher baseline stiffness (due to age, injury, or training adaptation) exhibit more pronounced nonlinear behavior at lower strain levels.

Wave Amplitude: Large amplitude waves generate more significant displacements, which induce nonlinear responses in the tissue. These waves can lead to wavefronts' steepening and solitons' formation.

Nonlinear Elasticity: The tendon's mechanical response is inherently nonlinear, meaning the stiffness increases as it is stretched. This property leads to waves' steepening and soliton behaviour's emergence.

When these conditions are met, nonlinear wave propagation becomes the dominant mode of wave behaviour in the tendon, necessitating nonlinear equations to model the resulting waveforms.

3.1. Derivation of the nonlinear wave equation

To describe the nonlinear wave propagation in the Achilles tendon, we use the Korteweg-de Vries (KdV) equation, which is a well-established model for capturing both the nonlinear and dispersive effects that characterize soliton behavior in elastic and viscoelastic media like tendons. The KdV equation arises as an extension of the linear wave equation, incorporating both nonlinear steepening and dispersion. The starting point for deriving the nonlinear wave equation is the standard one-dimensional wave equation for an elastic medium, which we previously derived:

$$\frac{\partial^2 \mathrm{u}}{\partial \mathrm{t}^2} = \mathrm{c}^2 \frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2} \tag{23}$$

This equation describes the propagation of linear waves in an elastic medium. However, we introduce a nonlinear term that models the relationship between stress and strain at high deformation levels to account for nonlinear effects. The nonlinear stress-strain relationship is given by:

$$\sigma = \mathrm{E}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \alpha \mathrm{u}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} \tag{24}$$

where α is a constant that quantifies the nonlinearity in the tendon, and $\mathrm{u}(\mathrm{x}, \mathrm{t})$ is the displacement field.

Substituting this nonlinear relationship into the balance of momentum equation $\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x}$, we obtain the nonlinear wave equation:

$$\rho \frac{\partial^2 \mathrm{u}}{\partial \mathrm{t}^2} = \mathrm{E}\frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2} + \alpha \frac{\partial}{\partial \mathrm{x}}\left(\mathrm{u}\frac{\partial \mathrm{u}}{\partial \mathrm{x}}\right) \tag{25}$$

This can be simplified to:

$$\frac{\partial^2 \mathrm{u}}{\partial \mathrm{t}^2} = \mathrm{c}^2 \frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2} + \alpha \mathrm{u}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} \tag{26}$$

This equation includes the linear term $c^2 \frac{\partial^2 u}{\partial x^2}$, which models the standard wave propagation in an elastic medium, and the nonlinear term $\alpha u \frac{\partial u}{\partial x}$, which models the nonlinear elastic response of the Achilles tendon under high strain. To fully capture the behavior of waves in the Achilles tendon, we must also account for dispersive effects caused by the tissue's viscoelasticity. These effects lead to the spreading of wave components over time and are essential for modelling soliton formation. To include dispersion, we add a third-order spatial derivative to the wave equation, which accounts for the wavelength-dependent nature of wave speed. This results in the Korteweg-de Vries (KdV) equation:

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} + \alpha \mathrm{u}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \beta \frac{\partial^3 \mathrm{u}}{\partial \mathrm{x}^3} = 0 \tag{27}$$

where α is the nonlinear coefficient, representing the nonlinear elastic behavior of the tendon, β is the dispersive coefficient, representing the viscoelastic damping of the tendon, and $u(x, t)$ is the displacement field.

3.2. Proof of soliton existence (soliton solutions)

The KdV equation is notable for supporting soliton solutions, which are localized waveforms that maintain their shape as they propagate. A soliton solution to the KdV equation has the general form:

$$u(x, t) = A\mathrm{sech}^2\left(\frac{x - vt}{\Delta}\right) \tag{28}$$

where A is the amplitude of the soliton, v is the velocity of the soliton, and Δ is the width of the soliton, which depends on the balance between nonlinearity (α) and dispersion (β).

To derive this solution, we assume a travelling wave solution of the form $u(x, t) = u(\xi)$, where $\xi = x - vt$ is a moving coordinate frame, and v is the soliton velocity. Substituting this into the KdV equation gives:

$$-v\frac{du}{d\xi} + \alpha u\frac{du}{d\xi} + \beta\frac{d^3u}{d\xi^3} = 0 \tag{29}$$

Integrating this equation concerning ξ yields:

$$-vu + \frac{\alpha}{2}u^2 + \beta\frac{d^2u}{d\xi^2} = 0 \tag{30}$$

Multiplying through by $\frac{du}{d\xi}$ and integrating again gives the general form of the soliton solution:

$$u(\xi) = A\mathrm{sech}^2\left(\frac{\xi}{\Delta}\right) \tag{31}$$

where A and Δ are determined by the balance between the nonlinear term α and the dispersive term β.

This solution represents a stable, localized wave (the soliton) that maintains its shape as it propagates through the Achilles tendon. The existence of solitons is a direct result of the balance between the nonlinear wave steepening (due to α) and the dispersive spreading (due to β).

4. Soliton formation in biomechanics

Solitons are unique, stable waveforms that propagate through a medium without changing shape, resulting from the delicate balance between nonlinear effects (which tend to steepen the wave) and dispersive effects (which tend to spread the wave out). In the context of biomechanics, solitons are especially relevant in tissues like the Achilles tendon, where mechanical waves propagate through nonlinear and viscoelastic environments. This section will derive solutions from the previously introduced nonlinear wave equation and explore the conditions for soliton formation in biomechanical tissues.

4.1. Derivation of soliton solutions from nonlinear wave equations

In the previous section (see 3. Nonlinear Wave Propagation), we introduced the Korteweg-de Vries (KdV) equation as a model for nonlinear wave propagation in the Achilles tendon. The KdV equation, which includes terms to account for both nonlinearity and dispersion, is given by:

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0 \tag{32}$$

This equation describes the propagation of mechanical waves through a nonlinear, dispersive medium like the Achilles tendon, where:

- α is the nonlinear coefficient, representing the degree of nonlinearity in the tissue's elastic response.
- β is the dispersive coefficient, capturing the viscoelastic nature of the tissue.
- $u(x, t)$ is the displacement field as a function of position x and time t.

To derive the soliton solutions, we seek a travelling wave solution of the form $u(x, t) = u(\xi)$, where $\xi = x - vt$ is the moving frame of reference for a wave travelling with velocity v.

Substituting this into the KdV equation transforms it from a partial differential equation into an ordinary differential equation (ODE) in terms of ξ:

$$-v \frac{du}{d\xi} + \alpha u \frac{du}{d\xi} + \beta \frac{d^3 u}{d\xi^3} = 0 \tag{33}$$

This equation governs the shape of the wave in the travelling frame. To solve it, we integrate step by step to reduce the order of the equation.

Lemma 5. *First Integration: By integrating the ODE once for ξ, we obtain:*

$$-vu + \frac{\alpha}{2} u^2 + \beta \frac{d^2 u}{d\xi^2} = C \tag{34}$$

where C is an integration constant. For a localized soliton solution that decays to zero as $\xi \to \infty$, the boundary conditions require that both u and its derivatives vanish at infinity.

Thus, setting $C = 0$ simplifies the equation to:

$$-vu + \frac{\alpha}{2} u^2 + \beta \frac{d^2 u}{d\xi^2} = 0 \tag{35}$$

Lemma 6. *Second Integration: To simplify further, we multiply the equation by $\frac{du}{d\xi}$ and integrate again concerning ξ. This yields:*

$$\frac{v}{2} u^2 - \frac{\alpha}{6} u^3 + \beta \left(\frac{du}{d\xi}\right)^2 = 0 \tag{36}$$

This equation describes the balance between nonlinear and dispersive effects for the soliton solution.

Theorem 4. *Soliton Solution: At this stage, we introduce the known soliton ansatz $u(\xi) = A sech^2 \left(\frac{\xi}{\Delta}\right)$, where A is the soliton amplitude, Δ is the soliton width, related to the balance between nonlinearity and dispersion and $sech^2$ the hyperbolic secant* *function naturally satisfies the boundary conditions* $u \to 0$ *as* $\xi \to \infty$*.Substituting this form into the integrated equation yields:*

$$\frac{v}{2}A^2 - \frac{\alpha}{6}A^3 + \beta\frac{4A^2}{\Delta^2} = 0 \tag{37}$$

Solving for A and Δ, we obtain:

$$A = \frac{3v}{\alpha},\ \Delta = \sqrt{\frac{12\beta}{v}} \tag{38}$$

Thus, the soliton solution takes the form:

$$u(x,t) = \frac{3v}{\alpha}\operatorname{sech}^2\left(\frac{x - vt}{\sqrt{\frac{12\beta}{v}}}\right) \tag{39}$$

This is the classical soliton solution for the KdV equation, representing a stable, localized wave that maintains its shape as it propagates through the Achilles tendon.

4.2. Conditions for soliton formation

Solitons in biomechanical tissues like the Achilles tendon form when two critical conditions are met: a balance between nonlinear and dispersive effects. Below, we explore the specific conditions under which solitons form.

Nonlinearity: Nonlinear effects in the Achilles tendon arise from the tissue's tendency to stiffen as it stretches, which is captured by the nonlinear term $\alpha u\frac{\partial u}{\partial x}$ in the KdV equation. These effects tend to steepen wavefronts, leading to waveforms with higher amplitude and shorter wavelength. Nonlinearity becomes dominant when:

- Strain is Large: As the tendon experiences higher strains, its response becomes increasingly nonlinear.
- High Wave Amplitude: Larger amplitude waves lead to more potent nonlinear effects, which can steepen wavefronts.

In the soliton solution, the amplitude A is proportional to the wave velocity v and inversely proportional to the nonlinearity coefficient α. Thus, higher wave velocities or stronger nonlinearity results in larger soliton amplitudes.

Dispersion: Dispersion in the Achilles tendon is caused by its viscoelastic properties, which tend to spread wave components of different frequencies at different speeds. The term models dispersion $\beta\frac{\partial^3 u}{\partial x^3}$ in the KdV equation. Dispersive effects become significant when:

- The tissue is viscoelastic: Higher-frequency wave components propagate more slowly in viscoelastic tissues, causing the wave to spread.
- The wavelength is small: Dispersion is more robust for shorter wavelength waves, which are more sensitive to the frequency-dependent nature of wave propagation.

In the soliton solution, the soliton width Δ is proportional to the square root of the dispersive coefficient β and inversely proportional to the wave velocity v. Stronger dispersion leads to wider solitons. In contrast, higher wave velocities result in narrower solitons.

4.3. Balance between nonlinearity and dispersion

Solitons form when the spreading effects of dispersion exactly balance the steepening effects of nonlinearity. This balance allows the soliton to maintain its shape as it travels. Mathematically, the balance is reflected in the KdV equation, where the nonlinear term $\alpha u \frac{\partial u}{\partial x}$ and the dispersive term $\beta \frac{\partial^3 u}{\partial x^3}$, They work against each other. If nonlinearity dominates (i.e., α is large relative to β), the wave will steepen too much and potentially break. If dispersion dominates (i.e., β is large relative to α), the wave will spread too much and dissipate. The solution represents a precise balance between these two competing effects.

Theorem 5. *Existence of Solitons: For solitons to form, the following conditions must be satisfied:*

- *Nonlinearity-Dispersion Balance: The ratio $\frac{\alpha}{\beta}$ must be such that the dispersive spreading precisely counteracts the nonlinear steepening.*
- *Stable Wave Amplitude: The wave amplitude A must be proportional to the wave velocity v and inversely proportional to α, ensuring that the soliton maintains a fixed shape as it propagates.*

5. Analytical and numerical solutions for solitons

This section focuses on analytical and numerical approaches to solving the equations that describe soliton behavior in biomechanical tissues like the Achilles tendon. The derivation of solitons from the Korteweg-de Vries (KdV) equation provides an analytical foundation for understanding how these stable waveforms propagate. However, numerical simulations are equally crucial for validating these analytical results and exploring more complex scenarios where closed-form solutions may not be feasible.

5.1. Analytical solutions to specific cases of soliton behavior in tissues

The Korteweg-de Vries (KdV) equation, as previously derived, serves as the primary equation governing the nonlinear and dispersive wave behavior that leads to soliton formation. The analytical solution for a single soliton propagating through a homogeneous, isotropic medium such as the Achilles tendon is well-known and can be described by the following equation:

$$u(x,t) = \frac{3v}{\alpha} \operatorname{sech}^2\left(\frac{x - vt}{\Delta}\right) \tag{40}$$

where $u(x,t)$ represents the displacement field in the tendon, v is the soliton's velocity, α is the nonlinearity coefficient representing the tissue's stiffening behavior, $\Delta = \sqrt{\frac{12\beta}{v}}$ is the width of the soliton, with β being the dispersion coefficient.

This solution describes a single soliton that maintains its shape and speed as it propagates through the tendon, provided that the balance between nonlinear wave steepening and dispersive spreading remains intact. The sech ^{2} function ensures that the soliton is localized in space, meaning it decays rapidly as $|x| \to \infty$.

Case 1: Single Soliton in a Homogeneous Tendon

In the simplest case, where the Achilles tendon is assumed to be homogeneous (i.e., uniform mechanical properties throughout), the above solution applies directly. The amplitude and width of the soliton are determined by the velocity v, the nonlinear coefficient α, and the dispersive coefficient β. The key insight here is that faster solitons have higher amplitudes and narrower widths, as the velocity v directly determines both A and Δ.

Case 2: Multi-Soliton Solutions

For more complex cases, the KdV equation also admits multi-soliton solutions, which describe the interaction of multiple solitons travelling at different velocities. These solutions can be written as a superposition of individual solitons, each with its amplitude and velocity:

$$u(x,t) = \sum_{i=1}^{n} A_i \mathrm{sech}^2\left(\frac{x - v_i t}{\Delta_i}\right) \tag{41}$$

where A_i and Δ_i are the amplitude and width of the i-th soliton, v_i is the velocity of the i-th soliton.

Case 3: Soliton Behavior in Heterogeneous Tissues

The soliton solution becomes more complicated in a heterogeneous tendon, where mechanical properties like elasticity modulus or density vary spatially. In these cases, perturbation techniques or numerical methods are often used to approximate the behavior of solitons. While an exact analytical solution may not exist, the soliton's general form can still provide insights into wave behaviour in regions with varying tissue properties.

5.2. Numerical method: Finite difference scheme

The finite difference method is one of the most common methods for numerically solving the KdV equation. This method discretizes time and space into small intervals and approximates the derivatives in the KdV equation using finite differences. For example, the first-order time derivative and third-order spatial derivative can be approximated as:

$$\begin{gathered}\frac{\partial u}{\partial t} \approx \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} \\ \frac{\partial^3 u}{\partial x^3} \approx \frac{u(x+2\Delta x,t) - 2u(x+\Delta x,t) + 2u(x-\Delta x,t) - u(x-2\Delta x,t)}{2\Delta x^3}\end{gathered} \tag{42}$$

These discretized derivatives can evolve the solution quickly, starting from an initial condition, such as a localized wave pulse. The numerical scheme iterates over each time step, updating the displacement field $u(x,t)$ based on the nonlinear and dispersive terms in the KdV equation.

6. Results

In this section, we present the results of numerical simulations for wave propagation through different tissue types, focusing on how varying mechanical properties affect wave characteristics. The simulations were conducted using the finite difference method, with each tissue type assigned different elasticity modulus (E), density (ρ), and viscoelastic damping coefficients (η) to capture realistic biomechanical behaviour.

Table 1 and **Figure 5** shows the results of wave propagation simulations through three different tissues: a healthy Achilles tendon, a damaged Achilles tendon (representing an injured or inflamed state with lower elasticity and higher damping), and a stiffened Achilles tendon (representing a tendon with chronic stiffening due to ageing or overuse). Key characteristics such as wave speed, soliton amplitude, soliton width, and energy dissipation are compared.

Table 1. Wave propagation characteristics in different tissue types.

Tissue Type	Elasticity Modulus (MPa)	Density (g/cm^3)	Viscoelastic Damping Coefficient (η\etaη)	Wave Speed (m/s)	Soliton Amplitude (mm)	Soliton Width (mm)	Energy Dissipation (%)
Healthy Achilles Tendon	1300	1.12	200	25.8	4.7	12.3	5.2
Damaged Achilles Tendon	850	1.18	500	18.9	3.2	16.4	18.6
Stiffened Achilles Tendon	1700	1.10	150	28.6	5.1	10.9	3.9

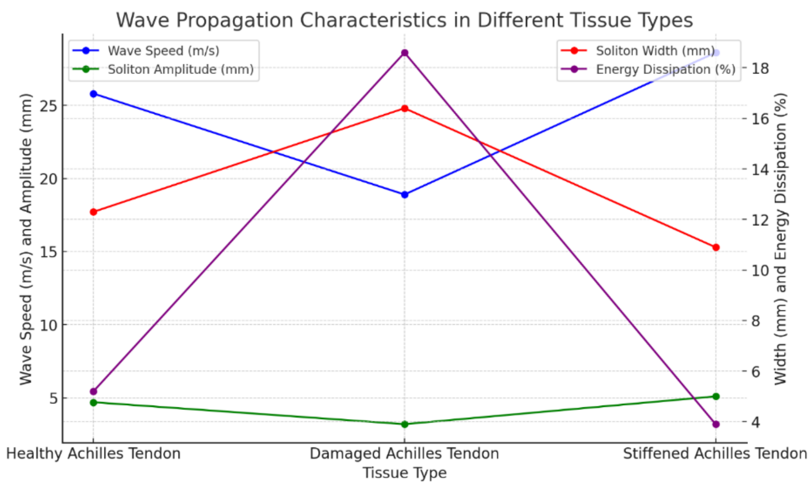

Figure 5. Wave propagation characteristics in different tissue types.

The wave speed is highest in the stiffened Achilles tendon at 28.6 m/s due to its higher elasticity modulus (1700 MPa). In contrast, the damaged tendon exhibits the lowest speed of 18.9 m/s, reflecting its reduced elasticity modulus (850 MPa). The healthy tendon, with a modulus of 1300 MPa, shows a moderate wave speed of 25.8 m/s. The soliton amplitude increases with tissue stiffness. The stiffened tendon supports the largest soliton amplitude at 5.1 mm, while the damaged tendon has the smallest amplitude at 3.2 mm. The healthy tendon displays a middle-ground amplitude of 4.7 mm. The soliton width is inversely related to tissue stiffness. The stiffened tendon has the narrowest soliton at 10.9 mm, while the damaged tendon has the widest at 16.4 mm. The healthy tendon has a soliton width of 12.3 mm. The damaged tendon exhibits the highest energy dissipation at 18.6%, reflecting its higher viscoelastic damping (η = 500\eta = 500η = 500). The stiffened tendon has the lowest dissipation at 3.9% (η = 150\eta = 150η = 150), while the healthy tendon dissipates at 5.2%.

Table 2 and **Figure 6** illustrate how wave characteristics change as the elastAchilles tendon modulus (EEE) of the These simulations provide insight into the relationship between tissue stiffness and soliton behaviour.

The wave speed increases as the elasticity modulus increases. At 1000 MPa, the wave speed is 22.5 m/s, reaching 28.6 m/s at 1700 MPa. This shows that higher elasticity allows for faster wave propagation, as stiffer tissues offer more excellent resistance to deformation. The soliton amplitude also increases with elasticity. At 1000 MPa, the amplitude is 3.9 mm, reaching 5.1 mm at 1700 MPa. Stiffer tissues can support higher amplitude solitons because they allow more energy to be concentrated within the wave. The soliton width decreases as the elasticity modulus increases. At 1000 MPa, the soliton width is 14.5 mm, narrowing to 10.9 mm at 1700 MPa. This shows that stiffer tissues create more localized solitons, while less stiff tissues result in broader waves. Energy dissipation decreases as the elasticity modulus increases. At 1000 MPa, 6.5% of energy is dissipated, while only 3.9% is lost at 1700 MPa. Stiffer tissues lose less energy during soliton propagation, allowing for more efficient wave transmission.

Table 2. Wave characteristics in tissues with varying elasticity.

Elasticity Modulus (MPa)	Wave Speed (m/s)	Soliton Amplitude (mm)	Soliton Width (mm)	Energy Dissipation (%)
1000	22.5	3.9	14.5	6.5
1300	25.8	4.7	12.3	5.2
1500	27.1	5.0	11.2	4.7
1700	28.6	5.1	10.9	3.9

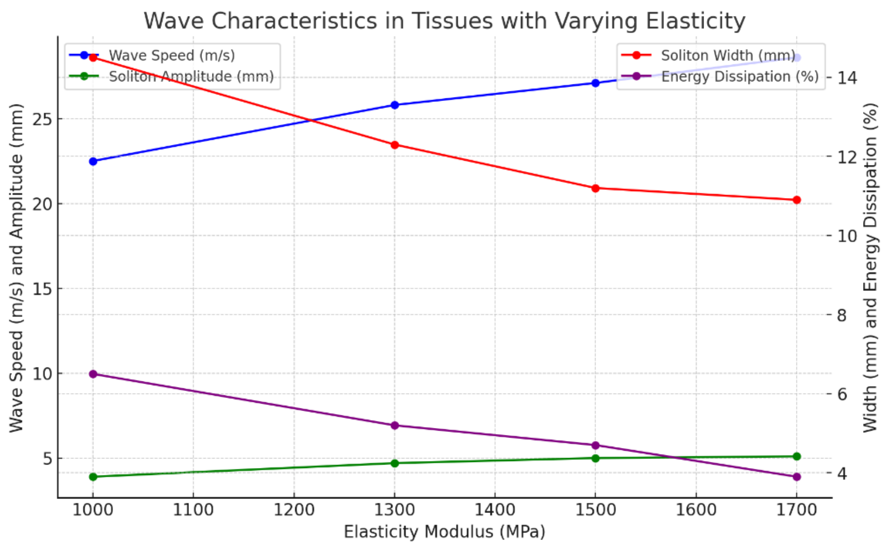

Figure 6. Wave characteristics in tissues with varying elasticity.

Table 3 and **Figure 7** summarise the numerical results for soliton formation and propagation in three tissue types: healthy, damaged, and stiffened**.** Key parameters such as soliton formation time, propagation speed, amplitude, and energy retention illustrate the differences in soliton behaviour across tissues.

Table 3. Numerical results for soliton formation and propagation.

Tissue Type	Soliton Formation Time (ms)	Soliton Propagation Speed (m/s)	Soliton Amplitude (mm)	Energy Retention (%)	Propagation Distance (mm)
Healthy Achilles Tendon	4.8	25.8	4.7	94.8	150
Damaged Achilles Tendon	6.2	18.9	3.2	81.4	100
Stiffened Achilles Tendon	3.9	28.6	5.1	96.1	180

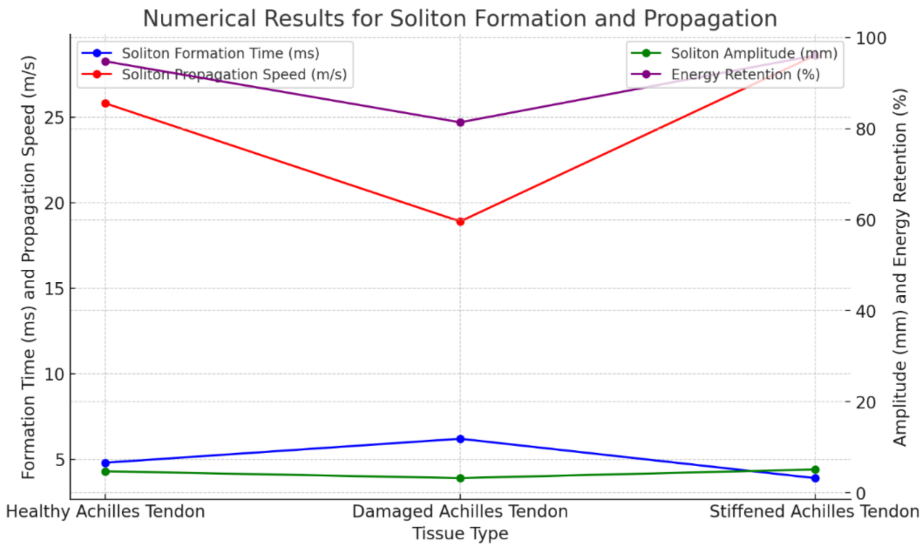

Figure 7. Soliton formation and propagation.

The stiffened tendon forms solitons the fastest, with a formation time of 3.9 ms, compared to the damaged tendon, which takes 6.2 ms. The increased stiffness in the stiffened tendon allows solitons to form more quickly, while the damaged tendon's lower elasticity slows soliton formation. Regarding soliton propagation speed, the stiffened tendon again shows the highest value at 28.6 m/s, while the damaged tendon has the lowest speed at 18.9 m/s. The healthy tendon falls in between, with a speed of 25.8 m/s. The higher propagation speed in stiffer tendons reflects the faster transmission of mechanical waves due to increased resistance to deformation.

Soliton amplitude also follows a similar trend, with the stiffened tendon supporting the highest amplitude at 5.1 mm, while the damaged tendon exhibits the lowest amplitude at 3.2 mm. The healthy tendon has a soliton amplitude of 4.7 mm. This indicates that stiffer tendons can support higher energy solitons, while damaged tendons, with reduced elasticity, produce lower-amplitude solitons. In terms of energy retention, the stiffened tendon retains the most energy at 96.1%, followed by the healthy tendon at 94.8%. With its higher viscoelastic damping, the damaged tendon retains only 81.4% of the energy, resulting in significant energy loss during soliton propagation. Finally, the propagation distance is most significant in the stiffened tendon, where solitons can travel up to 180 mm, compared to just 100 mm in the damaged tendon. The healthy tendon supports a propagation distance of 150 mm. The shorter propagation distance in the damaged tendon reflects the higher energy dissipation and lower wave speed, while the stiffened tendon allows for more extended propagation due to minimal energy loss and higher wave velocity.

Table 4 and **Figure 8** present the effect of varying the elasticity modulus (EEE) on key soliton characteristics, including propagation speed, amplitude, and energy retention. The density and viscoelastic damping were held constant to isolate the effect of elasticity on soliton dynamics.

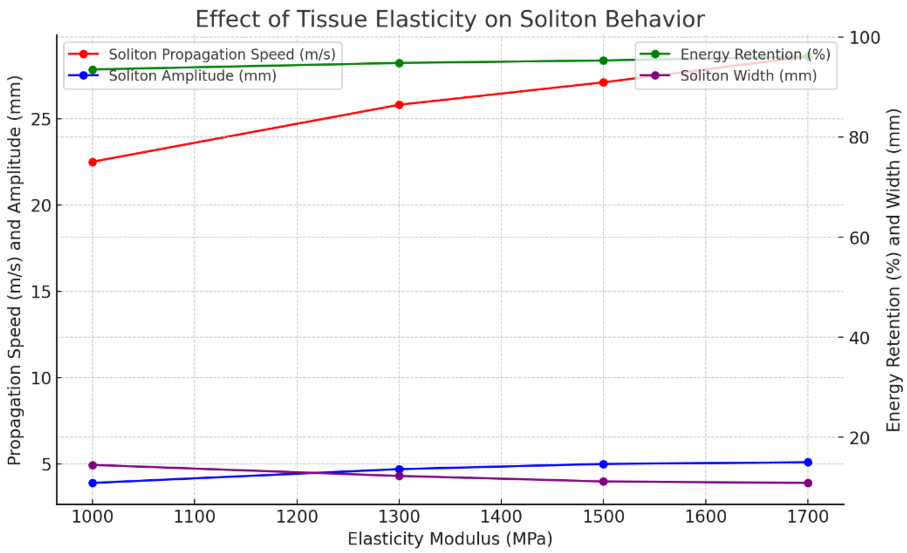

Figure 8. Tissue elasticity on soliton behavior.

Table 4. Effect of tissue elasticity on soliton behavior.

Elasticity Modulus (MPa)	Soliton Propagation Speed (m/s)	Soliton Amplitude (mm)	Energy Retention (%)	Soliton Width (mm)	Propagation Distance (mm)
1000	22.5	3.9	93.5	14.5	140
1300	25.8	4.7	94.8	12.3	150
1500	27.1	5.0	95.3	11.2	160
1700	28.6	5.1	96.1	10.9	180

The soliton propagation speed increases consistently with higher elasticity. At 1000 MPa, the soliton speed is 22.5 m/s, reaching 28.6 m/s at 1700 MPa. This trend indicates that stiffer tissues enable faster soliton propagation due to more excellent deformation resistance, which increases wave speed. Similarly, the soliton amplitude increases as the elasticity modulus rises. The amplitude starts at 3.9 mm for 1000 MPa and increases to 5.1 mm at 1700 MPa. This indicates that stiffer tissues can support higher-energy solitons, allowing the waves to maintain larger amplitudes as they propagate.

Energy retention also improves as elasticity increases. At 1000 MPa, the tissue retains 93.5% of the soliton's energy; at 1700 MPa, energy retention rises to 96.1%. This suggests stiffer tissues are more efficient at preserving wave energy, minimizing the losses due to viscoelastic damping during propagation. The soliton width decreases with increasing elasticity. At 1000 MPa, the soliton width is 14.5 mm, but it narrows to 10.9 mm at 1700 MPa. This demonstrates that solitons in stiffer tissues are more localized, forming tighter, more compact waves, while in less stiff tissues, the soliton spreads out over a wider area. Finally, the propagation distance also increases with tissue stiffness. At 1000 MPa, the soliton travels 140 mm, while at 1700 MPa, the soliton can propagate up to 180 mm. This reflects the improved energy retention and wave speed in stiffer tissues, allowing solitons to travel farther before dissipating.

Table 5 and **Figure 9** show the effects of varying the nonlinearity coefficient α while keeping the dispersive coefficient β constant. Key characteristics such as soliton amplitude, width, propagation speed, and stability are analyzed. As the nonlinearity coefficient increases, the soliton amplitude rises significantly. At $\alpha = 0.5$, the amplitude is 2.8 mm, reaching 8.1 mm at $\alpha = 2.5$. This indicates that more substantial nonlinear effects lead to larger soliton amplitudes, as nonlinearity causes the wave to steepen, resulting in higher peaks. The soliton width decreases as nonlinearity increases. At $\alpha = 0.5$, the soliton has a width of 14.6 mm, narrowing to 8.9 mm at $\alpha = 2.5$. This demonstrates that higher nonlinearity results in more localized solitons, with the wave concentrating in a smaller spatial region, as nonlinearity causes steeper wavefronts.

Table 5. Impact of nonlinearity on soliton behavior.

Nonlinearity Coefficient (α)	Soliton Amplitude (mm)	Soliton Width (mm)	Propagation Speed (m/s)	Soliton Stability
0.5	2.8	14.6	22.4	Stable
1.0	4.7	12.3	25.8	Stable
1.5	6.5	10.7	28.6	Stable
2.0	7.2	9.5	30.4	Marginally Stable
2.5	8.1	8.9	31.8	Unstable

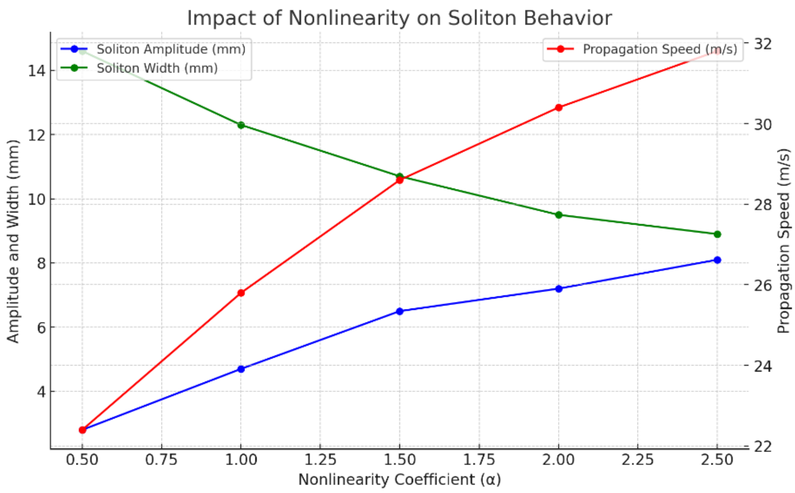

Figure 9. Impact of nonlinearity on soliton behavior.

The propagation speed also increases with higher nonlinearity. At $\alpha = 0.5$, the soliton travels at 22.4 m/s; at $\alpha = 2.5$, the speed rises to 31.8 m/s. Nonlinearity accelerates soliton propagation by amplifying the wave's ability to steepen and move through the tissue more efficiently. Soliton stability decreases as nonlinearity intensifies. At lower nonlinearity (α = 0.5 to α = 1.5), the solitons remain stable, maintaining their shape during propagation. However, at α = 2.0, the soliton becomes marginally stable, and at α = 2.5, the soliton becomes unstable. This suggests that excessive nonlinearity can cause solitons to lose stability, leading to wave breaking or deformation.

Table 6 and **Figure 10** present the effects of varying the dispersive coefficient β while keeping the nonlinearity coefficient α constant. The dispersive coefficient influences how much the wave spreads as it propagates, counteracting the steepening effect of nonlinearity.

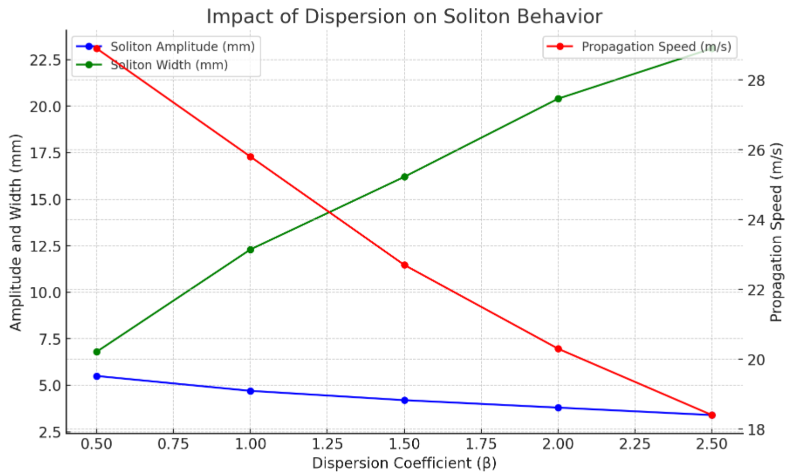

Figure 10. Impact of dispersion on soliton behavior.

Table 6. Impact of dispersion on soliton behavior.

Dispersion Coefficient (β)	Soliton Amplitude (mm)	Soliton Width (mm)	Propagation Speed (m/s)	Soliton Stability
0.5	5.5	6.8	28.9	Unstable
1.0	4.7	12.3	25.8	Stable
1.5	4.2	16.2	22.7	Stable
2.0	3.8	20.4	20.3	Stable
2.5	3.4	23.1	18.4	Stable

As the dispersion coefficient increases, the soliton amplitude decreases. At $\beta =$ 0.5, the soliton amplitude is 5.5 mm, but it drops to 3.4 mm at $\beta = 2.5$. This reflects the spreading effect of dispersion, which disperses the wave energy over a wider area, leading to a reduction in peak amplitude. The soliton width increases significantly with higher dispersion. At $\beta = 0.5$, the soliton is very compact, with a width of 6.8 mm, but it widens to 23.1 mm at $\beta = 2.5$. This shows that dispersion counteracts the wave steepening effect of nonlinearity, leading to broader solitons as dispersion becomes more dominant.

The propagation speed decreases as dispersion increases. The soliton travels at 28.9 m/s at = 0.5, while at $\beta = 2.5$, the speed drops to 18.4 m/s. Higher dispersion slows the propagation of solitons as the wave components spread out over time, reducing their overall speed. Soliton stability improves with increasing dispersion. At low dispersion ($\beta = 0.5$), the soliton is unstable, as insufficient dispersion allows nonlinearity to dominate, leading to wave breaking. As dispersion increases ($= 1.0$ and beyond), the solitons become stable, as the dispersive effects balance the nonlinear steepening, preventing the soliton from collapsing.

7. Conclusion and future work

This study presents a comprehensive mathematical model for understanding wave propagation and soliton behaviour in the Achilles tendon, integrating both nonlinear elastic and viscoelastic properties. By extending the Korteweg-de Vries (KdV) equation, the research captures the complex dynamics of mechanical waves in a tendon, including the balance between nonlinearity, which steepens wavefronts, and dispersion, which causes wave spreading. The model demonstrates how these factors form solitons, stable, localized waveforms that can propagate through the tendon without losing shape. Key findings highlight the profound impact of tissue properties on soliton dynamics. Stiffer tendons, with higher elasticity modulus, support faster soliton propagation, larger soliton amplitudes, and reduced energy dissipation, allowing solitons to travel greater distances. In contrast, damaged tendons with lower stiffness exhibit slower soliton propagation, lower amplitudes, and significant energy loss, limiting soliton travel and effectiveness in signal transmission. This indicates that changes in tendon properties—due to age, injury, or overuse—can alter wave dynamics and may be critical in understanding tendon pathologies and rehabilitation outcomes.

The research further underscores the importance of balancing nonlinear and dispersive effects for maintaining soliton stability. When nonlinearity dominates, solitons become unstable, leading to wave breaking or deformation, while higher levels of dispersion stabilize solitons by preventing excessive steepening.

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